An Extension of Nadler's Fixed Point Theorem

Oo Oo Khine*

Abstract

In this paper, the concepts of multi-valued contraction mappings on complete metric spaces are discussed with some illustrated examples. Nadler's multi-valued fixed point theorem is stated. The simpler proof of a generalization of Nadler's multi-valued fixed point theorem is established.

Key words: teachers' job satisfaction, teacher commitment

Introduction

In 1922, the mathematician, Stefan Banach, established a fixed point theorem known as the "Banach Contraction Principle" which is one of the most important results of Mathematical analysis and as the main source of metric fixed point theory. The Banach Contraction Principle has been generalized in many different directions. The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler in 1969. He proved multi-valued extension of the Banach contraction theorem. The Nadler's fixed point theorem for multi-valued contraction mappings has been extended in many directions. An extension of Nadler's fixed point theorem is given by Gordji, Baghani, Khodaei and Ramezani.

1. Definitions and Notations

In this section, we describe some definitions, some notations and some results will be used in later sections. Throughout this paper we denote by R the set of all real numbers, by N the set of natural numbers and by d the usual metric on the set X, i.e., $d(x, y) = |x - y|$ for all x, y $\in$X.

1.1 Definition. Let (X, d) be a metric space and $\{x_n\}$ be a sequence of X.

(i) $\{x_n\}$ is said to *converge or to be convergent* if for any $\varepsilon > 0$, there exists an x $\in$X such that $\lim_{n\to\infty} d(x_n, x) = 0.$ x is called the limit of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x.$

(ii) $\{x_n\}$ is said to be *Cauchy sequence* if for every $\varepsilon > 0$, there exists a positive integer N such that $d(x_m, x_n) < \varepsilon$ for every m, n > N.

(iii)(X, d) is *complete* if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

1.2 Definition. Let X and Y be nonempty sets. Then T: X $\to 2^Y$ is said to be a *multi-valued mapping*, where 2^Y denotes the collection of all non-empty subsets of Y.

1.3 Example. Let X = {1, 2, 3} and T: X $\to 2^Y$ be defined such that

$$T(x) = \begin{cases} \{1, 2\}, & x = 1 \\ \{3\}, & x = 2, 3. \end{cases}$$

Then T is a multi-valued mapping.

1.4 Remark. Every single-valued mapping T: X $\to$ Y can be represented as a multi-valued mapping if we define T: X $\to 2^Y$ by Tx = {T(x)}.

1.5 Example. Let X = {1, 2 , 3}, Y = {2, 4, 6, 8} and let T : X $\to$ Y be defined by T(x) = 2x for all x$\in$ X. Define T: X $\to 2^Y$ by Tx = {T(x)}. That is, T1 ={T(1)} = {2}, T2 ={T(2)} ={4} and T3 = {T(3)} ={6}. Then T is a multi-valued mapping.

1.6 Definition. Let (X, d) be a metric space and A $\subset$ X. Then *the distance of a point x*$\in$ X *to the set A* is defined by d(x, A) = $\inf_{y\in A}\{d(x,\ y) \mid x\in X\}$.

1.7 Example. Let (X, d) be a metric space. Let X = R and A = [1, 4], B =[8, 9]. Then the distance of 5$\in$X to A, d(5, A) = $\inf_{y\in[1,4]}\{d(5,\ y) \mid 5\in X\} = \inf_{y\in[1,4]}\{|5-y|\}=1$ and the distance of 5 $\in$X to B, d(5, B) = $\inf_{z\in[8,9]}\{d(5,\ z) \mid 5\in X\} = \inf_{z\in[8,9]}\{|5-z|\}=3$.

1.8 Definition. Let (X, d) be a metric space and CB(X) denotes the collection of all non-empty closed and bounded subsets of X. For A, B $\in$ CB(X), define

$$H(A,B)=\max\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\}.$$

It is known that H is a metric on CB(X), called the *Hausdorff metric* induced by the metric d.

1.9 Example. Let X = R, A = [0, 3] and B = [5, 7]. Then

$$\begin{aligned} H(A,B) &= \max\{\sup_{x\in A} d(x,\ B), \sup_{y\in B} d(y,\ A)\} \\ &= \max\{\sup_{x\in[0,3]} d(x, [5,7]),\ \sup_{y\in[5,7]} d(y,\ [0,3])\} \\ &= \max\{5,\ 4\} \\ &= 5. \end{aligned}$$

1.10 Definition. The mapping T: X $\to$ CB(X) is *continuous* if $u_n \to u$ implies H(Tu_n, Tu) $\to$ 0 for all sequences {u_n} in X.

1.11 Definition. Let X be any non-empty set. A point x_0 in X is said to be *a fixed point* of the multi-valued mapping T: X $\to 2^X$ if $x_0 \in Tx_0$.

1.12 Example. Let X = {1, 2, 3}. Let T: X $\to$ CB(X) be such that

$$Tx=\begin{cases}\{2\}, & x=1\\ \{1\}, & x=2\\ \{3\}, & x=3,\end{cases}$$

It can be seen that 1$\notin$T1, 2$\notin$T2 and 3$\in$T3. Thus 1 and 2 are not fixed points of T but 3 is a fixed point of T.

2. Multi-valued contraction mappings

In this section, we discuss the notion of multi-valued contraction mapping from the metric space (X, d) into the all closed and bounded subsets of X, CB(X).

2.1 Definition. Let (X, d) be metric space. A mapping T: X $\rightarrow$ CB(X) is said to be *multi-valued contraction mapping* if H(Tx, Ty) $\leq$ r d(x, y) for all x, y$\in$X and r $\in$ [0, 1).

2.2 Example. Let (X, d) be a metric space and let X = R. Let T: X $\rightarrow$ CB(X) defined by $Tx = [0, \frac{x}{3}]$ for all x$\in$ X.

For any x, y$\in$ R, and let x < y,

$$H(Tx, Ty) = H\left(\left[0, \frac{x}{3}\right], \left[0, \frac{y}{3}\right]\right)$$

$$= \max\{\sup_{u\in[0,\frac{x}{3}]} d(u, [0, \frac{y}{3}]), \sup_{v\in[0,\frac{y}{3}]} d(v, [0, \frac{x}{3}])\}$$

$$= \max\{0, \left|\frac{x}{3} - \frac{y}{3}\right|\}$$

$$= \left|\frac{x}{3} - \frac{y}{3}\right|$$

$$= \frac{1}{3}|x - y|$$

$$< \frac{1}{2} d(x, y)$$

Since there exists $r = \frac{1}{2} \in [0,1)$ such that H(Tx, Ty) < r d(x, y), it follows that T is a multi-valued contraction mapping.

2.3 Remark.

1. If (X, d) is a complete metric space then (CB(X), H) is a complete metric space.
2. Let (X, d) be metric space. A multi-valued contraction mapping T:X$\rightarrow$CB(X) is continuous mapping.

3. Nadler's Fixed Point Theorem

In this section, we first introduce the definition of the contraction mapping on a metric space and the Banach Contraction Principle, known as Banach's fixed point theorem.

3.1 Definition. Let X be a metric space. A mapping T: X $\rightarrow$ X is said to be *a contraction* on X if there is a real number r $\in$[0, 1) such that for all x, y $\in$X

$$d(T(x), T(y)) \leq r\, d(x, y).$$

3.2 Theorem. (Banach's Fixed Point Theorem). Let (X, d) be a complete metric space, then each contraction map f: X $\rightarrow$ X has a unique fixed point.

The following lemma is useful to proof the Nadler's fixed point theorem.

3.3 Lemma. Let (X, d) be a complete metric space and A, B$\in$CB(X). Then for each x$\in$A and r > 0 there exists y$\in$B such that

$$d(x, y) \le H(A, B) + r.$$

The following example is illustrated the Lemma 3.3.

3.4 Example. Let X = R, A = [a – t, a + t], B = [b – s, b + s] and 0 < t ≤ s, where a, b ∈ R.

Let d be defined by d(a, b) = $|a-b|$ for all a, b ∈ R.

It can be seen that A, B ∈CB(X) and thus

$$\begin{aligned} H(A,B) &= \max\{\sup_{x\in A} d(x,\ B), \sup_{y\in B} d(y,\ A)\} \\ &= \max\{d(b-s,\ a-t),\ d(b+s, a+t)\} \\ &= \max\{|(b-s)-(a-t)|, |(b+s)-(a+t)|\} \\ &= \max\{|(b-a)-(s-t)|, |(b-a)+(s-t)|\} \\ &\ge |(b-a)-(s-t)| \\ &\ge |b-a| - |s-t| \\ &= d(a,b)-(s-t) \end{aligned}$$

So, d(a, b) ≤ H(A, B) + r, where r = s – t.

The following result is a generalization of the Banach Contraction Principle, known as Nadler's fixed point theorem.

3.5 Theorem. Let (X, d) be a complete metric space. If T: X → CB(X) is a multi-valued contraction mapping, then T has a fixed point.

4. An extension of Nadler's fixed point theorem

The following result is a generalization of Nadler's fixed point theorem. The simpler proof given below is almost exactly the same as the proof in the following reference.

4.1 Theorem. Let (X,d) be a complete metric space and let $T: X \to CB(X)$ such that

$$H(T_x, T_y) \le \alpha\ d(x,y) + \beta\left[d(x,T_x) + d(y,T_y)\right] + \gamma\left[d(x,T_y) + d(y,T_x)\right]$$

for all x, y ∈ X, when $\alpha, \beta, \gamma \ge 0$ and $\alpha + 2\beta + 2\gamma < 1$.

Then T has a fixed point.

Proof. Let $u_0 \in X$ and Let $u_1 \in Tu_0$. Let $r = \dfrac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}$. If r = 0, then the proof is completed.

Now assume that r > 0. Since $Tu_0, Tu_1 \in CB(X)$ and $u_1 \in Tu_0$, by lemma (3.3), there exists $u_2 \in Tu_1$ such that

$$d(u_1, u_2) \le H(Tu_0, Tu_1) + r$$

Again, since $Tu_1, Tu_2 \in CB(x)$ and $u_2 \in Tu_1$, by lemma $(4,3)$, there exists $u_3 \in Tu_2$ such that

$$d(u_2, u_3) \le H(Tu_1, Tu_2) + r^2$$

Continuing this iterative process, in general , there exists $u_{n+1} \in Tu_n$ for each $n \in N$ such that

$$d(u_n, u_{n+1}) \le H(Tu_{n-1}, Tu_n) + r^n.$$

Thus,

$$\begin{aligned} d(u_n, u_{n+1}) &\le H(Tu_{n-1}, Tu_n) + r^n \\ &\le \alpha\, d(u_{n-1}, u_n) + \beta\left[d(u_n, Tu_n) + d(u_{n-1}, Tu_{n-1})\right] + \gamma\left[d(u_n, Tu_{n-1}) + d(u_{n-1}, Tu_n)\right] + r^n \\ &\le \alpha\, d(u_{n-1}, u_n) + \beta\left[d(u_n, u_{n+1}) + d(u_{n-1}, u_n)\right] + \gamma\left[d(u_{n-1}, u_n) + d(u_n, u_{n+1})\right] + r^n \end{aligned}$$

for all $n \in N$. It follows that

$$\begin{aligned} d(u_n, u_{n+1}) &\le r\, d(u_{n-1}, u_n) + \frac{r^n}{1-(\beta+\gamma)} \\ &\le r^2 d(u_{n-2}, u_{n-1}) + \frac{2r^n}{1-(\beta+\gamma)} \\ &\le \ . \quad . \quad . \\ &\le r^n d(u_0, u_1) + \frac{nr^n}{1-(\beta+\gamma)} \end{aligned}$$

Then we have,

$$\begin{aligned} \sum_{n=0}^{\infty} d(u_n, u_{n+1}) &= d(u_0, u_1) + d(u_1, u_2) + d(u_2, u_3) + \dots \\ &\le d(u_0, u_1) + \left\{ rd(u_0, u_1) + \frac{r}{1-(\beta+\gamma)} \right\} + \left\{ r^2 d(u_0, u_1) + \frac{2r^2}{1-(\beta+\gamma)} \right\} + \dots \\ &\qquad + \left\{ r^n d(x_0, x_1) + \frac{nr^n}{1-(\beta+\gamma)} \right\} \\ &= d(u_0, u_1)\left[1 + r + r^2 + \dots\right] + \left[r + 2r^2 + 3r^3 + \dots\right] \frac{1}{1-(\beta+\gamma)} \\ &= d(u_0, u_1)\left[\frac{1}{1-r}\right] + \left[r + 2r^2 + 3r^3 + \dots\right] \frac{1}{1-(\beta+\gamma)} \\ &= \frac{d(u_0, u_1) + \left[r + r^2 + r^3 + \dots\right]\left[\frac{1}{1-(\beta+\gamma)}\right]}{1-r} \\ &= \frac{d(u_0, u_1)}{1-r} + \frac{r}{(1-r)^2}\left[\frac{1}{1-(\beta+\gamma)}\right] \\ &< \infty \quad \text{since } r < 1. \end{aligned}$$

Thus, $\{u_n\}$ is a Cauchy sequence.

Since X is a complete metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Now we have to show that x^* is a fixed point of T.

$$\begin{aligned} d(x^*,Tx^*) &\le d(x^*,x_{n+1}) + d(x_{n+1},Tx^*) \\ &\le d(x^*,x_{n+1}) + H(Tx_n,Tx^*) \\ &\le d(x^*,x_{n+1}) + \alpha\, d(x_n,x^*) + \beta\left[d(x_n,Tx_n) + d(x^*,Tx^*)\right] + \gamma\left[d(x_n,Tx^*) + d(x_{n+1},x^*)\right] \end{aligned}$$

for all $n \in N$. Therefore,

$$d(x^*,Tx^*) \le d(x^*,x_{n+1}) + \alpha\, d(x_n,x^*) + \beta\left[d(x_n,x_{n+1}) + d(x^*,Tx^*)\right] + \gamma\left[d(x_n,Tx^*) + d(x_{n+1},x^*)\right]$$

for all $n \in N$. By taking the limit $n \to \infty$, then we have

$$d(x^*,Tx^*) \le (\beta+\gamma)\, d(x^*,Tx^*).$$

On the other hand $\beta+\gamma<1$, then $d(x^*,Tx^*)=0$. It follows that $x^* \in Tx^*$.

Hence, x* is a fixed point of T. ■

Conclusion

Many authors contributed the extended results of the Nadler's fixed point theorem. This paper is one of the extended results of the Nadler's fixed point theorem.

Acknowledgements

I would like to express my great gratitude to Rector Dr. Aye Aye Myint and Pro-rectors Dr. Pyone Pyone Aung and Dr. Kay Thway Hlaing, Yangon University of Education, for giving me permission to write this paper. I would also like to express my gratitude to my Professor, Professor Dr. Lwin Mar Htun, Head of Department, Mathematics Department for her support and encouragement to carry out the research work.

References

C. B. Nadler, (1969) "Multi-valued contraction mappings", Pacific J. Math. 30, 475- 488.

E. Kreyszig,(1978) *Intorductory Functional Analysis with Applications,* John Wiley and Sons. Inc.

I. A. Rus, (2001) *Generalized Contractions and Applications*, Cluj Univercity Press, Cluj-Nappa.

J. M. Joseph, E. Ramganesh, (2013) "Fixed Point Theorem on Multi-valued Mappings", Int. J. Math. Analysis, 1, no. 2, 123- 127.

M. E Gordji, H. Baghani, H. Khodaei and M. Ramezani, (2010) A Generalization of Nadler's Fixed Point Theorem, J. Nonlinear Sci Appli.3, Vol.2, 148- 151.